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Reciprocal Schrödinger Equation: Durations of Delay and Formation of States in Scattering Processes

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Abstract Reciprocal Schrödinger equation for scattering matrix $\partial S(\omega, \mathbf{r})/i\partial\omega = \hat{\tau}(\omega, \mathbf{r})S(\omega, \mathbf{r})$ determines temporal function, its real part presents the Wigner–Smith duration of delay and imaginary part describes the duration of resulted (dressed) state formation. "Deduction" of this equation is executed by the Legendre transformation of classical action function with subsequent transition to quantum description and, in the covariant form, by a temporal variant of the Bogoliubov variational method. Temporal functions are expressed via propagators of fields, they are formally equivalent to adding a photon line of zero energy-momentum to the Feynman graphs. As an apparent example they can be clearly interpreted in the oscillator model via polarization and conductivity of medium. It is shown that the adiabatic hypothesis in scattering theory represents an implicit account of temporal parameters. By these functions are described some renormalization procedures, their physical sense is refined, etc.

Keywords Temporal functions · Scattering · Dressing · Sense of renormalization

1 Introduction

Temporal characteristics of process of scattering should include, in principle, two types of magnitudes: duration of a delay of colliding particles during their interaction and duration of formation (dressing) of products of reaction. Their research has an uneasy history: in the beginning of development of quantum theory McColl had shown that calculation of duration of tunnel transition conducts to negative value [1, 2]. Therefore during long time was factually accepted that any estimations of temporal characteristics, besides the uncertainty principles, are practically not necessary or impossible.

The situation should be principally changing when Frank introduced in the theory of Ĉerenkov radiation a notion of path length (or duration), necessary for a gradual formation

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of real photon by "a superluminal in media" electron [3]. Without such concept was incomprehensible the discrete character of this emission, and Frank had been forced to estimate the interference picture of continuously emitted (virtual) waves that can lead to the real emission of single photons at resonance conditions.

However this theory had been remaining without further researches and developments onto many years. The first, as far as I know, quantum investigation in this direction has been made and published by Moshinsky [4–6]. He had calculated, through the non-stationary Schrödinger equation, the time duration needed for establishment of the definite state of electron after its transition onto upper level with some damped oscillations ("temporary diffraction"), i.e. the duration of resulted state formation.

The first (semi-qualitative) consideration of time delay in processes of tunneling had been performed, as far as I know, by Bohm [7]. Then Ter-Mikaelyan [8] and Landau and Pomeranchuk [9] had considered the duration of photon formation in the theory of bremsstrahlung: it is the time duration needed for a virtual coat formation around particle, its dressing (the reviews [10–12]).

The more constructive and physically more transparent magnitude of time delay under an elastic scattering was introduced by Wigner [13] through the partial phase shifts, $\tau_l(\omega) = d\delta_l/d\omega$, generalized by Smith [14–18] via S-matrix as

$$\tau_1(\omega) = \mathrm{Re}(\partial/i\partial\omega)\ln S = (\partial/\partial\omega)\arg S. \tag{1.1}$$

Then Goldberger and Watson had deduced on the base of (1.1) a "coarse-grain" Schrödinger equation by which the generality of this definition had been shown [19]. But at their approach the magnitude (1.1) had been introduced artificially, by the serial decomposition of Fourier transformed response function $S(t)$ of linear relation,

$$O(t) = S(t) \otimes I(t) = \int dt' S(t'-t) I(t'), \tag{1.2}$$

or its logarithm near the selected frequency without discussion of its imaginary part, higher terms and dependence on space variables.

Another approach, which seems at first glance distinctive from the Wigner–Smith one, was suggested by Baz' [20] for consideration of nonrelativistic tunneling processes: to a scattering particle is attributed magnetic moment and its rotation at the scattering process is analyzed (the method of "Larmor clocks").

After these initial investigations a number of various definitions of duration of scattering processes and interaction was offered, different determinations of duration of interactions are introduced, e.g. the reviews collected in [21].

Our purpose in the series of papers [22–38] was to reveal that functions describing duration of scattering processes and formation of new states are present in already existing theories: if these concepts reflect essential features of a reality they should be found in the theories, which adequately describe several experiments.

And really, it has been revealed that the temporal functions can be found out in relativistic dispersion relations [22–25]; they are naturally manifesting out at summation of indefinite perturbative series of multiphoton processes as the opportunity of capture of the following photon is determined by the duration of virtual keeping of previously captured energy by scatterer, and therefore they determine thresholds of new processes [26–29]. Temporal functions are directly connected with propagators of particles and it explains why the calculations without their direct introduction are possible [30]. Thereby these functions do

not need to be entered artificially, ad hoc; a more careful investigation of existing theories instead it is possible.

On the other hand, however, it was necessary to show that the direct use of temporal functions at analyses, at least, of some processes has practical and not only a gnosiological sense. On their basis the theory of optical dispersion has been constructed [31, 32], some features of phase transitions are established [33–37], the opportunity of the "nonlocality in the small", i.e. instantaneous tunneling jumps of excitations within the scope of near field, is shown [38].

The last possibilities are connected with the definite duration of state formation ("dressing") expressed as

$$\tau_2 = \mathrm{Im}(\partial \ln S / i\partial\omega) = (\partial/\partial\omega) \ln |S|. \tag{1.3}$$

As far as I know, the similar expression for τ_2 was introduced, for the first time, by Pollak and Miller [39, 40] and was interpreted as the duration of tunneling process. (Note that the duration of particle formatting can be calculated by dynamical considerations, it presents the special direction in the high energy physics, cf. the review [41].)

As it is proven in [38], superluminal transfer of excitations (jumps) through a linear passive substance can be affected by nothing but by the instantaneous tunneling of virtual particles. The tunneling distance $c|\tau_2|$ is expressed via the deficiency in the energy relative to the nearest stable (resonance) state $\Delta(\hbar\omega)$ as the relation of uncertainty type:

$$\tau_2 \Delta\omega = \pi, \tag{1.4}$$

the nonlocality of the electromagnetic field must be described by the 4-potential A_μ, whereas the fields **E** and **B** remain unconnected to the near field. (The overview of this theorem is given below; it can explain, in particular, the paradox of tunneling calculations [1, 2].)

Nevertheless the existence of two independent expressions (1.1) and (1.3) even logically seems unsatisfactory: delay at scattering and duration of formation should be in some way or other interrelated. Formally they can be combined as $\tau_1(\omega, \mathbf{r}) + i\tau_2(\omega, \mathbf{r}) = (\partial / i\partial\omega) \ln S(\omega, \mathbf{r})$, which appears as an equivalent of the equation

$$\partial S(\omega, \mathbf{r}) / i\partial\omega = \tau(\omega, \mathbf{r}) S(\omega, \mathbf{r}), \tag{1.5}$$

as though it is an analog of the Schrödinger equation for S-matrix, rewritten via the transformation of $t \leftrightarrow \omega$ type and with a "temporal" operator $\tau(\omega, \mathbf{r})$ instead Hamiltonian.

Such is indeed the case: *this problem and further investigation of temporal functions on this basis is the purpose of the paper*.

The direct derivation of (1.5), which would establish the foothold in the duration problems, is performed by some equivalent of the Legendre transformation of Schrödinger equation (Sect. 2). The main properties of unified temporal functions $\tau(\omega, \mathbf{r})$ and their interrelations with the uncertainty magnitudes are considered in Sect. 3. As these functions are causal, the dispersion relations and corresponding sum rules for them can be established that demonstrates some principal properties of temporal functions (Sect. 4). The received results are discussed in Sect. 5 on an example of the simplest oscillator model of medium that descriptively reveals the physical sense of both temporal functions.

If temporal parameters can be considered as the results of interference of waves coming from different points, it seems that the suiting functions for their comparative investigation should be the Wigner functions (Sect. 6). Their consideration shows that the expressions of temporal functions are close to propagators (Green functions), and it will be proven in Sect. 7 in the frame of formal theory of scattering.

In Sect. 8 temporal functions and their covariant forms will be considering by the methods of quantum field theory and the equations of (1.5)-type will be generalized till completely covariant analogues of Tomonaga–Schwinger equations. It allows to show that the offered theory can be considered as the justification of the adiabatic hypothesis of quantum field theory and its generalization, the revealing of physical sense of such formal, as seems, mathematical procedure.

Section 9 is devoted to some problems of QED. Their considerations are continued in Sect. 10 by interpretation of renormalization procedures, the Pauli–Villars and the subtraction methods, and, more generally, the renormalization group equations via temporal functions.

In the Conclusions the main results are summed up and some perspectives of further investigations are mentioned.

2 Legendre Transformation

The basic equation of quantum dynamics, $i\partial S/\partial t = \mathbf{H}S$, can be formally "deduced" from the Hamilton–Jacobi equation for classical action function,

$$(\partial/\partial t)S_{cl}(q_i; \partial S_{cl}/\partial q_i; t) = H_{cl}(q_i; \partial S_{cl}/\partial q_i; t), \tag{2.1}$$

by the Schrödinger-type heuristic substitution:

$$S_{cl} \to i\hbar \ln\{S(t, \mathbf{r})/\hbar\} \tag{2.2}$$

and replacement of classical variables x, p by corresponding operators ($c = \hbar = 1$ below). Notice that this substitution mathematically means the transition from the class $\mathfrak{L}_1$ of integrable functions to the class of $\mathfrak{L}_2$ functions, to the Hilbert space.

The transition to new variables in the classical action function is achieving by the Legendre transformation:

$$S_{cl}(q'; p'; t') = S_{cl}(q; p; t) - \sum(q'q + p'p + t't). \tag{2.3}$$

The canonical transformation from the time variable t to the energy variable, $t \to t' = H \to E$, in the equation (2.1) results in

$$S_{cl}(t; \ldots) - Ht = S_{cl,L}(E; \ldots), \tag{2.4}$$

and the canonical equation (2.1) is transformed into the temporal Hamilton–Jacobi equation:

$$(\partial/\partial E)S_{cl,L}(E; \ldots) = -T_{cl}(E; \ldots), \tag{2.5}$$

in which the role of Hamiltonian plays a (classical) function of duration of considered process. It leads to classical temporal Hamilton equations and so on.

The analog of Schrödinger-type substitution (2.2),

$$S_{cl,L}(E; \ldots) \to i \ln S(E, \mathbf{r}), \tag{2.6}$$

leads to the quantum equation (more familiar for such notations symbol ω instead E is used):

$$\partial S(\omega, \mathbf{r})/i\partial\omega = \tau(\omega, \mathbf{r})S(\omega, \mathbf{r}), \tag{2.5$'$}$$

from which the determination of temporal function in accordance with the Legendre transformation follows:

$$\tau(\omega, \mathbf{r}) = \partial \ln S(\omega, r)/i\partial\omega. \tag{2.7}$$

These expressions conform to (1.5), but their "deduction" allows discussion of equation features and some its generalizations.

The Legendre transformation can be performed at nonzero values of the Hessian, i.e. the determinant of second derivatives:

$$J(t \to \omega) = (\ln S)_{tt}(\ln S)_{\mathbf{rr}} - ((\ln S)_{t\mathbf{r}})^2 \neq 0. \tag{2.8}$$

Its rewritten form,

$$J(t \to \omega) = (\partial H/\partial t)\overrightarrow{\nabla}\mathbf{p} - (\overrightarrow{\nabla} H)^2 \neq 0, \tag{2.8$'$}$$

evidently determines processes for which an introduction of the temporal functions $\tau(\omega, \mathbf{r})$ has physical sense. Note that as the Legendre transformation $\widehat{L}$ is performed by the involution operator, $\widehat{L}^2 = 1$, this transformation does not change magnitudes of observables and commutation relations.

Notice that the variation of function $\tau(\omega, \mathbf{r})$ immediately leads to the Fermat principle.

Further Legendre transformation $\mathbf{r} \to \mathbf{r}' = \mathbf{k}$ of the function $S(E, \mathbf{r})$ leads to the relation:

$$\rho(\omega, k) = i\partial \ln S(\omega, \mathbf{k})/\partial\mathbf{k}, \tag{2.9}$$

which must describe an space interaction region in dependence on energy-momentum. Operators $(\tau, \rho) = (\partial/i\partial\omega, i\partial/\partial\mathbf{k})$ form the 4-vector corresponding to the equation:

$$(\tau^2 - \rho^2 - s^2)S(\omega, \mathbf{k}) \equiv -(\partial_\omega^2 - \partial_\mathbf{k}^2 + s^2)S(\omega, \mathbf{k}) = 0 \tag{2.10}$$

with a 4-interval s. It can be considered as the reciprocal one to the Klein–Gordon equation, its general integral representation with the restrain $s^2 \geq 0$ leads to the relativistic generalizations of Kramers–Kronig dispersion relations [22–25].

The direct integration of (2.7) leads to the representation

$$S(\omega, \mathbf{r}) = S_0(\omega_0, \mathbf{r}) \exp\left[-i\int_{\omega_o}^{\omega} \tau(\omega, r)d\omega\right], \tag{2.11}$$

corresponding integration of (2.9) will be expressed via density of 4-volume of interaction.

It seems beneficial and interesting, for some brightening of physical sense of used substitutions, to recall, on one example, the possible role of temporal functions in classics and their correspondence with quantum functions (more precisely it may be a correspondence between theories described by functions of the $\mathfrak{L}_1$ and $\mathfrak{L}_2$ classes of integration).

So, the duration of rotation in the classical mechanics can be determined as

$$T = 2m\int_a^b dx/p(x), \quad p(x) = [2m(E - V(x))]^{1/2}, \tag{2.12}$$

p is the linear momentum of rotating particle, E and V are the complete and potential energies. By introducing the phase integral or the action function $A = 2\int_a^b p(x)dx$ the duration of process can be determined as $T = 4\partial A/\partial E$. The duration, for example, of a packet

spreading over a system of equidistant levels was determined in the "old" quantum mechanics [42] as

$$\Delta T \sim 1/(\partial \Delta E/\partial A) \approx 2\partial^2 A/\partial E^2. \tag{2.13}$$

In accordance with the Schrödinger heuristical principle the transition from classical mechanics into quantum one must be carried out by replacement of action function onto its logarithm: $A \to i\hbar \ln(S/\hbar)$, and just this substitution leads to the definition (2.2).

3 Some Features of Temporal Functions

The general solution (2.11) of (2.7) can be presented as

$$S(\omega, \mathbf{r}) = S_1(\omega_0, \mathbf{r}) \exp\left[i \int^{\omega} \tau_1(\omega, \mathbf{r}) d\omega - \int^{\omega} \tau_2(\omega, \mathbf{r}) d\omega \right], \tag{3.1}$$

where lower limits of integrals do not depend on ω; the functions τ_1 and τ_2 represent, correspondingly, (1.1) and (1.3). The unitarity of $S(\omega, \mathbf{k})$ allows to conclude, with the consideration of Cauchy- Schwartz inequality for (3.1)

$$|S(\omega, \mathbf{k})|^2 \equiv 1 = \left| \int S(\omega, \mathbf{r}) e^{i\mathbf{kr}} dr \right|^2 \le \int |S_0(\omega_0, \mathbf{r})|^2 d\mathbf{r} + \int \exp\left[-2 \int \tau_2(\omega, \mathbf{r}) d\omega \right] d\mathbf{r}, \tag{3.2}$$

that $\tau_2(\omega, \mathbf{r})$ cannot retain the constant sign over all frequencies interval. Its alternating may show an incompleteness of response function in the given space point. It can be *assumed* that the details of processes leading to the terminating of reaction, that are usually named as the particles (states) dressing, must be described just by the function τ_2.

At a simplified estimation (3.2) can be presented as

$$|S(\omega)| \simeq |S(\omega_0)| \exp[-(\omega - \omega_0)\tau_2], \tag{3.2′}$$

then the opportunity of Fourier-transformation of $S(\omega, \mathbf{r})$, i.e. the existence of the response function $S(t, \mathbf{r})$, dictates for the considered theory the inequality $(\omega - \omega_0)\tau_2 \le 0$ near maximal value of τ_2. It shows that at $\omega < \omega_0$ the duration of formation τ_2 may be negative, i.e. in the certain frequencies range the advanced emission or even superluminal phenomena are not excluded . Just such situation has place at a superluminal transfer of excitation and, as it had been shown in [38], corresponds to a lot of experimental data [43–46].

Addition of the following term to the $S(\omega, \mathbf{r})$ decomposition of (3.2′) type,

$$\sigma(\omega, \mathbf{r}) \equiv (\partial / i\partial\omega)^2 \ln S(\omega, \mathbf{r}) = -i\tau'(\omega, \mathbf{r}), \tag{3.3}$$

at the inverse Fourier transformation with $\tau_2 > 0$ leads to the "normal" response function:

$$S^{(+)}(t, \mathbf{r}) = S(\omega_0)(8\pi\sigma)^{-1/2} \exp[-i\omega_0 t - (t-\tau)^2/2\sigma]\{1 - \mathrm{erf}((t-\tau)/\sqrt{2\sigma})\}. \tag{3.4}$$

Thus (3.3) shows the broadening of signals on their path.

With $\tau_2 < 0$ such transformation results in the "anomalous" response function $S^{(-)}(t, \mathbf{r})$, which will be distinguished by the sign of errors function.

Thus, the complete response function is represented as the sum

$$S(t) = \theta(\tau_2) S^{(+)}(t) + \theta(-\tau_2) S^{(-)}(t), \tag{3.5}$$

which can be examined as an analogue of decomposition of the causal propagator $\Delta_c(x) = \theta(t)\Delta^{(-)} + \theta(-t)\Delta^{(+)}$, where $\Delta^{(\pm)}$ propagators correspond to positive and negative frequencies parts.

Let's consider some peculiarities of temporal functions connected with the uncertainty principle by the general method offered by Mandelstam and Tamm [47, 48].

Their consideration begin with the comparison of two quantum expressions for Hermitian operators **A** and **H**, the Hamiltonian, with the standard deviations ΔA and ΔH:

$$\Delta H \cdot \Delta A = \frac{1}{2}|\langle \mathbf{HA} - \mathbf{AH}\rangle|; \tag{3.6}$$

$$\hbar(\partial / i\partial t)\langle \mathbf{A}\rangle = \langle \mathbf{HA} - \mathbf{AH}\rangle, \tag{3.7}$$

which lead together to the equation

$$\Delta H \cdot \Delta A = \frac{1}{2}\hbar|\partial_t \langle \mathbf{A}\rangle|. \tag{3.8}$$

For its analysis they introduce the projector P of some definite state ψ_0:

$$P(t) = (\psi_0, \psi)\psi_0, \qquad P^2 = P, \qquad \langle P\rangle \leq 1, \tag{3.9}$$

its standard deviation is defined as

$$\Delta P(t) = (\langle P^2\rangle - \langle P\rangle^2)^{1/2} \equiv (\langle P\rangle - \langle P\rangle^2)^{1/2}. \tag{3.10}$$

The substitution of (3.10) in (3.8) is offering the main relation:

$$\Delta H \cdot (\langle P\rangle - \langle P\rangle^2)^{1/2} = \frac{1}{2}\hbar|\partial_t \langle \mathbf{P}\rangle|. \tag{3.11}$$

This differential equation with the initial value $P(0) = 1$ leads to the solution:

$$P(t) = \cos^2(\Delta H t/\hbar). \tag{3.12}$$

Therefore for decay processes with the halftime $\tau_{1/2}$, when $P(\tau_{1/2}) = 1/2$, it gives the Mandelstam–Tamm form of the energy-duration uncertainty relation:

$$\Delta H \cdot \tau_{1/2} = \pi\hbar/4. \tag{3.13}$$

But for the process of excitation transfer, when at the completion of transferring there appears a new stable state and therefore $P(\tau_2) = 1$, it gives

$$\Delta H \cdot \tau_2 = n\pi\hbar, \quad n = 1, 2, \ldots, \tag{3.14}$$

where both magnitudes ΔH and τ_2 must be simultaneously positive or negative. It completely conforms with all observations of superluminal transfer of excitations in the anomalous dispersion regions, at FTIR and so on and completely corresponds to (1.4) at $n = 1$.

In order to describe the effects of nonlocality, all phenomena with $\tau_2 < 0$ can or even must be considered as the observation of virtual processes that are mathematically very near to a description of instantons. Moreover, since the described instant jumps are represented by virtual exchanges, such states can be considered as the quasiparticles with their own

peculiarities and sizes, within which signals of definite frequency intervals are instantly transferable. Hence such formations can be conceived as the instanton-type quasiparticles.

The value of (3.14) exceeds the common Heisenberg limit. The possibility of exceeding can be, in general, substantiated by the most general formal deduction of uncertainty relations given by Schrödinger [49]. His deduction brings to such expression for standard deviations:

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle \mathbf{AB}-\mathbf{BA}\rangle|^2+\frac{1}{4}[\langle \mathbf{AB}+\mathbf{BA}\rangle-2\langle \mathbf{A}\rangle\langle \mathbf{B}\rangle]^2, \tag{3.15}$$

which differs from the more usual form by the last term and can strengthen the condition of (3.14) type. The Heisenberg limit of this expression, with $|\langle \mathbf{AB}-\mathbf{BA}\rangle| \to \hbar$ and omitting of the second term, shows a minimal value of uncertainties, which can be achieved for pure states, in the weakly correlated conditions. But this limit can be exceeded for some physical magnitudes (compare [50], such possibilities are mentioned in recent investigations also, e.g. [51, 52] and references therein).

Notice also the deduction of uncertainty principle with the operator $\partial/i\partial\omega$ by Wigner [53–55]. In this article was specially underlined that such uncertainties can be separately considered over different axes, this peculiarity can be a starting point at investigation of phenomena of FTIR.

4 Dispersion Relations and Sum Rules

Response functions in $(\omega, \mathbf{r})$-representations are governed by the temporal equation and simultaneously they satisfy the Kramers–Kronig dispersion relations as it requires the principle of causality:

$$S_c(\omega)=\frac{1}{\pi i}\int_{-\infty}^{\infty} d\eta \frac{S_c(\eta)}{\omega-\eta} \tag{4.1}$$

(we write them in the simplest form with $S(\omega=0)=0$, here and below singular integrals are taken via the Cauchy principal values). This duality allows obtaining some principal conclusions.

By differentiation of (4.1) or by its substitution into (1.5) these dispersion relations can be represented in two forms:

$$\tau(\omega)S_c(\omega)=-\frac{1}{\pi}\int_{-\infty}^{\infty} d\eta \frac{S_c(\eta)}{(\omega-\eta)^2}, \tag{4.2}$$

$$\tau(\omega)S_c(\omega)=\frac{1}{\pi i}\int_{-\infty}^{\infty} d\eta\tau(\eta) \frac{S_c(\eta)}{\omega-\eta}. \tag{4.2'}$$

Equating of their right sides leads to the sum rule:

$$\int_{-\infty}^{\infty} d\omega S_c(\omega)\frac{1}{\omega}\left[\tau(\omega)-\frac{i}{\omega}\right]=0. \tag{4.3}$$

This expression can be satisfied, in particular, with the equalities

$$\tau_1(\omega)=0, \qquad \tau_2(\omega)=1/\omega, \tag{4.4}$$

which show that even at the absence of delay for formation of the out state (wave or particle, etc.) the certain time duration, twice bigger the common uncertainties value, is needed.

Notice that at the Fourier transformation of the temporal equation (1.5) such expression follows:

$$tS(t) = \int_0^\infty dt' S(t')\tau(t-t'), \quad t \geq 0, \tag{4.5}$$

and if $\lim tS(t) = 0$ at $t \to 0$, the sum rule of (4.3) type

$$\int dt S(t)\tau(-t) = 0 \tag{4.5'}$$

can be deduced. Further derivatives of the equation (1.5) lead to more complicate sum rules, by checking of which the singularities of $S(t)$ at $t \to 0$ can be determined.

The temporal functions $\tau(t)$ must be causal, i.e. $\tau(t)$ must be zero at $t < 0$ and therefore it obeys the equation of constraints [22–25]:

$$\tau(t) - \tau(0) = \theta(t)[\tau(t) - \tau(0)], \tag{4.6}$$

$\theta(t)$ is the Heaviside step function and its uncertainty at $t = 0$ requires corresponding subtractions that can contain, in principle, higher derivatives of $\tau(0)$ also. But in absence of tunneling and instant transitions [38] these subtractions can be omitted and for temporal functions the new dispersion relations can be written:

$$\tau(\omega) = \frac{1}{\pi i}\int_{-\infty}^{\infty} d\eta \frac{\tau(\eta)}{\omega - \eta}, \tag{4.6'}$$

which evidently interconnect $\tau_1(\omega)$ and $\tau_2(\omega)$. They are compatible, in particular, with the conditions (4.4) and with representations of these functions via propagators in Sect. 6.

The analyticity of causal $S(\omega)$ allows its representing as the Bläschke product:

$$S(\omega) = \text{const} \cdot \omega^{-p} \prod_n \frac{\omega - \omega_n + i\gamma_n/2}{\omega - \omega_n - i\gamma_n/2}. \tag{4.7}$$

With taking into account the relations (4.6') the sum rule (4.3) can be rewritten via an interaction operator $T(\omega) = S(\omega) - 1$ as:

$$\int_{-\infty}^{\infty} d\omega T(\omega)\frac{1}{\omega}\left[\tau(\omega) - \frac{i}{\omega}\right] = 0. \tag{4.3'}$$

Since $\omega^p S(\omega)$ is the meromorph function, the substituting of $T(\omega)$ into this equality and closing the integration contour in the upper half-plane produces the representation:

$$\tau(\omega) = \sum_n [\omega - \omega_n + i\gamma_n/2]^{-1} \pm ip/\omega, \quad p > 0. \tag{4.8}$$

Temporal functions have physical sense for positive frequencies, for negative frequencies they are determined by the analytical continuation: $\tau(-\omega) = \tau^*(\omega)$, which follows the analyticity of $S(\omega)$.

It allows the determination of Fourier transforms:

$$\tau(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega e^{i\omega t}\frac{\partial}{\partial\omega}\ln S(\omega) = \text{res}\sum_n \exp(i\omega_n t), \tag{4.9}$$

the last equality follows the meromorphity of (4.7) at integer p. It leads to representations:

$$\tau_1(t) = -\sum_n \cos(\omega_n t)\exp(-\gamma_n|t|); \qquad \tau_2(t) = -i\,\mathrm{sgn}(t)\tau_1(t), \tag{4.10}$$

i.e. temporal functions are represented by a set of damped oscillators with self-frequencies modulated by the widths of transmission bands, index n numerates self-values.

The oscillator character ("time diffraction") of the expressions (4.10) and modulation of oscillations by self-band widths must be underlined (cf. [4–6]). Note that at the standard approach the duration of processes is usually taken as $1/\gamma$, without account of their oscillating character.

The analyticity of $S(\omega + i\varsigma)$ in the upper half-plane allows to write such integral over the closed contour:

$$\oint \tau(\omega)d\omega = \oint \tau_1(\omega)d\omega = 2\pi(N - P), \tag{4.11}$$

where N and P are zeros and poles of temporal function into the contour. Poles of $\tau_1(\omega)$ signify impossibility of signal transferring on these frequencies through the system (frequencies locking) or particles capture at scattering processes. Zeros show that corresponding signals are passed through system without delays, etc. Really (4.11) represents a variant of the Levinson theorem of quantum scattering theory, e.g. [19].

The maximum-modulus principle for $S(\omega)$ shows that as $\tau_2(\omega)$ is determined via its derivative, it can not be equal to zero at any frequency: the formation of outgoing signal (wave, particle, state) always requires some temporal duration.

It represents the main physical result of this section.

5 Illustrative Example: Harmonic Oscillator

Let's illustrate some of obtained results by consideration of the simplest model, the oscillator with damping of (4.8)-type:

$$x'' - \gamma x' + \omega_0^2 x = f(t). \tag{5.1}$$

The complete causal solution of (5.1) can be written via the Green functions:

$$x(t) = \int_{-\infty}^{t} dt' G(t - t')f(t'); \qquad G(t) = G_0(t) + G_1(t), \tag{5.2}$$

and (5.2) can be considered as a model description of (3.1). The response part of the complete Green function is the solution of non-homogeneous equation, the Fourier image of which is

$$G_1(\omega) = -1/2\pi[(\omega + i\gamma/2)^2 - \omega_1^2] \tag{5.3}$$

with $\omega_1^2 = \omega_0^2 - \gamma^2/4$.

The corresponding causal temporal functions are:

$$\tau_1(\omega) = \gamma/2[(\omega - \omega_1)^2 + \gamma^2/4] + \{\omega_1 \to -\omega_1\}, \tag{5.4}$$

$$\tau_2(\omega) = (\omega - \omega_1)/2[(\omega - \omega_1)^2 + \gamma^2/4] + \{\omega_1 \to -\omega_1\}. \tag{5.5}$$

The last expression shows the possibility of advanced or superluminal propagation at $\omega < \omega_1 - \gamma^2/8\omega_1$ (cf. [38] and superluminal transfer in macroscopic oscillator systems [56]).

Apart from some exotic cases $\gamma \ll \omega_0$ and at $|\omega - \omega_0| > \gamma$ and then with $\gamma \to 0$ it can be taken that

$$\tau_1(\omega) \simeq \gamma/2[(\omega - \omega_0)^2 + \gamma^2/4] \to \pi\delta(\omega - \omega_0), \tag{5.4′}$$

$$\tau_2(\omega) \simeq (\omega - \omega_0)/2[(\omega - \omega_0)^2 + \gamma^2/4] \to 1/(\omega - \omega_0), \tag{5.5′}$$

this shows the proximity of last expression to the uncertainty values.

It seems that the most evident and close to the intuitive physical representation of temporal functions may give their description in the Lorentz model of dispersive and absorbing media (e.g. [57]), where all media are represented as the set of oscillators with damping. Each oscillator is described by the Green function (5.3) with corresponding factor depending on density of scatterers and so on.

The real parts of dielectric susceptibility and conductance are expressed in this model, respectively, as

$$\varepsilon_1(\omega) - 1 \approx \omega_p^2(\omega - \omega_0)/2[(\omega - \omega_0)^2 + \gamma^2/4]; \tag{5.6}$$

$$\sigma_{\text{electr}}(\omega) \approx \omega_p^2/8\pi\gamma[(\omega - \omega_0)^2 + \gamma^2/4], \tag{5.7}$$

ω_p is the plasma frequency.

The comparison of (5.6-7) and (5.4-5), excluding the immediate vicinity of resonance, shows the possibilities of approximations:

$$\varepsilon_1(\omega) - 1 \approx (\omega_p^2/2\omega)\tau_2(\omega), \tag{5.6′}$$

$$\sigma_{\text{electr}}(\omega) \approx (\omega_p^2/4\pi\gamma^2)\tau_1(\omega). \tag{5.7′}$$

These relations give the evident interpretation of both temporal functions. So, the polarization of media is reasonably determined by duration of wave formation. And, it is also intuitively evident, the electrical conductivity, as (every) transfer process, is determined via the durations of EM waves delay, which can be induced by virtual momentum transferring to charged particles, i.e. by their movements in the direction of EM flux.

The more general connection of temporal functions with characteristics of media can be established in such fashion. The principle of entropy grows requires execution of the strong inequality for almost transparent passive dispersive media: $\partial(\omega\varepsilon)/\partial\omega \geq 0$ [58]. With the substitution $S \to \varepsilon(\omega) - \varepsilon(\infty) = \varepsilon_1 + i\varepsilon_2$, i.e. by the equation $\partial\varepsilon/\partial\omega = i\tau\varepsilon$, the real part of this general inequality can be rewritten as

$$\tau_2 \leq 1/\omega - \tau_1\varepsilon_2/\varepsilon_1. \tag{5.8}$$

At sufficiently low frequencies $\varepsilon_2 = (4\pi/\omega)\sigma_{\text{electr}}$, and this inequality is reducing to the simplest form:

$$\tau_1 + \tau_2 \leq 1/\omega, \tag{5.9}$$

which evidently shows that τ_2 can be negative in some frequencies region. In particular it must be negative in the region of anomalous dispersion, where must be expected a discordance between maxima of ε_1 and ε_2 [38], but for their description more realistic models are needed.

6 Temporal Wigner Functions

The Wigner functions describe the overlapping of space domains of states [59] as

$$w(\mathbf{k};\mathbf{r};t)=\left(\frac{1}{2\pi}\right)^3\int d\mathbf{q}e^{i\mathbf{qk}}\psi(\mathbf{r}-\mathbf{q}/2;t)\psi^*(\mathbf{r}+\mathbf{q}/2;t), \tag{6.1}$$

and are covariantly generalized as

$$w(k;x)=\left(\frac{1}{2\pi}\right)^4\int dve^{-ivk}\psi(x-v/2)\psi^*(x+v/2), \tag{6.2}$$

with 4-vectors k, x, v that describe the time-space overlapping (interference) of the quantum self-states [60]. Their quantum field interpretation via the creation and destruction operators descriptively shows that the interference of oppositely shifted wave functions in (6.2) must sum the maps of their possible variation onto 4-intervals.

Let us consider the one-particle temporal Wigner functions as the special case of (6.2),

$$w^{(+)}(\omega,t;\mathbf{r})=\frac{1}{2\pi}\int_0^\infty d\tau e^{i\omega\tau}\psi(t-\tau/2;\mathbf{r})\psi^*(t+\tau/2;\mathbf{r}); \tag{6.3}$$

$$w^{(-)}(\omega,t;\mathbf{r})=w^{(+)}(-\omega,t;\mathbf{r}). \tag{6.3$'$}$$

These functions evidently describe the temporary overlap of wave functions at one space point and therefore just these functions should characterize the time delay at collision process and the duration of states formation (space arguments will be hereafter omitted). Notice the proximity of such description to the Frank conjecture in the theory of Ĉerenkov radiation [3].

By time shifts of wave functions with the Hamiltonian $\mathbf{H}$,

$$\begin{aligned}\psi(t-\tau/2)&=\psi(t)\exp(i\mathbf{H}\tau/2);\\ \psi^*(t+\tau/2)&=\exp(i\mathbf{H}\tau/2)\psi^*(t),\end{aligned} \tag{6.4}$$

the temporal Wigner function (6.3) is rewritten as

$$w^{(+)}(\omega,t)=\psi(t)\delta_+(\omega-\mathbf{H})\psi^*(t)\to\psi(t)W^{(+)}(\omega,t)\psi^*(t). \tag{6.5}$$

These functions are the self-functions of the operator equation

$$\frac{\partial}{i\partial\omega}w^{(+)}(\omega,t)=i(\omega-E)^{-1}w^{(+)}(\omega,t) \tag{6.6}$$

of (1.5) type, E is the (complex) energy of system, $\mathbf{H}\psi=E\psi$. This equation can be considered as the reciprocal one to the Liouville equation in Schrödinger representation. It shows that the durations of scattering processes and of states formation should be described as the self-values of corresponding Green operators.

It must be noted that in distinction from the space Wigner functions the temporal functions are non-symmetric relative to their variables and therefore their self-values can be complex ones. It just corresponds to possibilities of retarded and advanced interactions.

Slightly another derivation of such equation can be examined on transition to the Heisenberg representation,

$$\psi(t-\tau/2) = \exp(i\omega\hat{\tau})\psi(t/2)\exp(-i\omega\hat{\tau}),$$
$$\psi^{+}(t+\tau/2) = \exp(i\omega\hat{\tau})\psi^{+}(t/2)\exp(-i\omega\hat{\tau}), \tag{6.4′}$$

with a temporal operator $\hat{\tau}$. Equation (6.6) can be rewritten as

$$-i\partial_{\omega}w^{(+)}(\omega,t) = \exp(i\omega\hat{\tau})[\hat{\tau}, Q]\exp(-i\omega\hat{\tau}), \tag{6.6′}$$

with function

$$Q(\omega) = \frac{1}{2\pi}\int_0^{\infty} dt e^{i\omega\tau}\psi(-\tau/2)\psi^{+}(\tau/2).$$

This representation naturally leads to the Hamilton equations for temporal operators.

The function (6.3), just as all Wigner functions, can be rewritten via the conjugate variable, via the energy,

$$-i\partial_{\omega}w^{(+)}(\omega,t;\mathbf{r}) = \frac{1}{2\pi}\int_0^{\infty} d\eta e^{i\eta t}\psi(\omega-\eta/2;\mathbf{r})\psi^{+}(\omega+\eta/2;\mathbf{r}). \tag{6.7}$$

Therefore the state formation and so on can be considered as a gradual process of energy alteration till their definite values for physical ("dressed") particles. This property can be evidently generalized on interactions of arbitrary number of particles. In a similar way may be considered the gradual evolution of (establishment in) other particles characteristics in scattering processes.

It can be noted, in particular, that if it is possible to introduce the operator of complete momentum $\mathbf{K}$, the Wigner functions in the close analogy with all above can be symbolically written as

$$w(\mathbf{k};\mathbf{r}) = \psi(\mathbf{r})\delta(\mathbf{k}-\mathbf{K})\psi^{+}(\mathbf{r}); \tag{6.8}$$

i.e. via the vector Green functions. (This possibility will not be considered here further.)

7 Formal Theory of Scattering

Inasmuch temporal functions can be represented via propagators, let's consider on this base the process of elastic scattering:

$$A + B \to A + B. \tag{7.1}$$

The kinetics of interaction must be described by the operator $\mathbf{S} = 1 - i\mathbf{T}$, where $\mathbf{T}$ is the operator of interaction, expressed via propagators $G(E) = (E - \mathbf{H})^{-1}$ and $g(E) = (E - \mathbf{H}_0)^{-1}$, the complete Hamiltonian $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$, self-values of the Hamiltonians are complex, $\mathbf{H}\psi = (E + i\Gamma)\psi$ and $\mathbf{H}_0\psi_0 = (E_0 + i\Gamma_0)\psi_0$, where Γ_0 and Γ are the natural and complete widths of the upper level (we shall restrict our consideration to two-level system).

As it was shown in [30] the duration of scattering and duration of new state formation are naturally expressed via propagators with account and without account of this interaction:

$$\Delta\hat{\tau} \equiv \hat{\tau} - \hat{\tau}_0 = i[G(E) - g(E)], \tag{7.2}$$

where $\hat{\tau}$ and $\hat{\tau}_0$ denote temporal characteristics of particle's complete way with and without interaction.

The differentiation of operator of interaction, $\mathbf{T} = \mathbf{V}/(1 - g\mathbf{V})$, with taking into account the determination $G = g + g\mathbf{V}G$ and the definition (7.2), leads to an equation

$$\partial\mathbf{T}/i\partial E = \Delta\hat{\tau}\mathbf{T}. \tag{7.3}$$

Under the transition to energy surface, $E = E(\mathbf{p})$, the matrix element of (7.2),

$$\langle\mathbf{p}|\Delta\hat{\tau}|\mathbf{p}\rangle = \pm i\sum\{[E - E_n - i\Gamma_n/2]^{-1} - [E - E_n^{(0)} - i\Gamma_n^{(0)}/2]^{-1}\}, \tag{7.4}$$

clearly shows its properties. So, $iG(E)$ can be interpreted as the time duration needed for particles flight with their elastic scattering and $ig(E)$ corresponds to the free transfer only.

Transfer in (7.2) into the coordinate representation,

$$G(\mathbf{r}) - g(\mathbf{r}) = -\frac{1}{(2\pi)^3}\int d\mathbf{p}\langle\mathbf{p}|\Delta\hat{\tau}|\mathbf{p}\rangle e^{i\mathbf{pr}}, \tag{7.5}$$

demonstrates the similarity of our definition with the Smith derivation of time delay at scattering processes [7].

Notice that such expression for the temporal operator also follows (7.2):

$$\Delta\hat{\tau} = ig\mathbf{V}G. \tag{7.6}$$

This allows, in particular, the expansion of temporal functions into the series of free Green functions and interaction vertices:

$$\Delta\hat{\tau} = ig\mathbf{V}g + ig\mathbf{V}g\mathbf{V}g + \cdots, \tag{7.7}$$

natural for quantum theories and useful for interpretations of these processes via Feynman graphs, etc. These forms show that the measurement of time characteristics of process is equivalent to addition of graphs of process by specific vertices (we shall return to this interpretation below).

The third form of temporal operator, which follows (7.2), can be expressed as

$$\Delta\hat{\tau} = ig\mathbf{T}g. \tag{7.6}$$

Its matrix element,

$$i\langle\mathbf{p}|\mathbf{T}|\mathbf{p}\rangle/[(E - E_0(\mathbf{p}))^2 + \Gamma_0^2(\mathbf{p})/4], \tag{7.7}$$

with the substitution of the known expression of scattering amplitude on the angle zero, $f(\mathbf{p},\mathbf{p}) = 4\pi^2 m\langle\mathbf{p}|\mathbf{T}(\mathbf{p})|\mathbf{p}\rangle$, and the transforming to energy surface $E = E(\mathbf{p})$ leads to the expression:

$$\langle\mathbf{p}|\Delta\hat{\tau}|\mathbf{p}\rangle = \frac{1}{2\pi^2 im\Gamma^2}f(\mathbf{p},\mathbf{p}). \tag{7.8}$$

The real part of (7.8) can be expressed, with taking into account the optical theorem of scattering theory, via the total cross-section of scattering:

$$\tau_1(\mathbf{p}) = \frac{p}{(2\pi)^3 m\Gamma^2}\sigma_{\text{tot}}(\mathbf{p}). \tag{7.9}$$

Notice that this result clarifies the great delay with the beginning of examination of temporal characteristics of scattering processes: the most part of this information is contained in the Green functions and cross-sections.

If we determine the (maximal) volume of interaction as $V = \sigma_{\max} u \tau_{\max}$, where u is the velocity of particle, $\tau_{\max} = 2/\Gamma$ and $\sigma_{\max}$ is the resonance cross-section, the mean value of duration of interaction can be determined as the balance relation,

$$\bar{\tau}_1(\mathbf{p})/\sigma_{\text{tot}}(\mathbf{p}) = \tau_{\max}/\sigma_{\max}. \tag{7.10}$$

For the practically most important optical region $\Gamma \sim 10^8\ \text{s}^{-1}$, $\sigma_{\max} = 4\pi/k^2$, $\sigma_{\text{tot}} = (4\pi/k)r_0$, $r_0 = e^2/mc^2$. Therefore for $k = 6.3 \times (10^4\text{–}10^3)\ \text{cm}^{-1}$ the relation (7.10) leads for nonresonant frequencies to

$$\bar{\tau}_1(\mathbf{p}) \sim (k/\Gamma)r_0 = 1.6 \times (10^{-16}\text{–}10^{-15})\ \text{s}, \tag{7.11}$$

this evidently does not contradict the usual representations of light propagation speed.

This magnitude allows an estimation of the mean value of index of refraction in nonresonant region. As it had been shown in [31, 32] the optical dispersion in a transparent, at least, region can be considered as the kinetic process of photons transfer through media. Such transfer must be described by the free path lengths $\ell = 1/N\sigma_{\text{tot}}$ with the vacuum velocity c, N is the density of outer (optical) electrons, and by the subsequent delays at each scatterer for the mean time (7.11). Thus the complete time, needed for photons transfer on a distance L, is equal to

$$\overline{T} = (L/c) + (L/\ell)\tau_1. \tag{7.12}$$

This estimation leads to the group velocity $u = L/T$ and, for nonresonant cases, to the group refractive index

$$n_{gr} \equiv \frac{c}{u} = \frac{c\overline{T}}{L} = 1 + cN\sigma_{\text{tot}}\tau_1 \sim 1 + N\frac{4\pi c}{\Gamma}r_0^2 \sim 1 + 3 \times 10^{-22}N, \tag{7.13}$$

which qualitatively corresponds to observations (N is of order of the Löschmidt number).

It must be underlined that the representation of temporal functions via propagators supports the results of Sect. 6: their analytical properties and the existence of dispersion relations of Kramers–Kronig type.

8 Duration of Interaction and Adiabatic Hypothesis

Let's try to reveal that the magnitudes of duration of interaction are implicitly contained in the standard theory in the form of adiabatic hypothesis. This hypothesis asserts that for the correct quantum calculations of transition amplitude such artificial substitution for the Hamiltonian is needed:

$$V(t) \to V(t)\exp(-\lambda|t|) \tag{8.1}$$

with switches to the limit $\lambda \to 0$ after all calculations (e.g. [19]).

Stueckelberg proposed more general approach to these problems via the causality condition [60]. Bogoliubov generalized his method by introduction of operations of "the switching interaction on and off" performed by some function $q(x) \in [0, 1]$, which characterizes the intensity of interaction: in the space-time regions with $q(x) = 0$ interaction is completely

absent and with $q(x) = 1$ is completely turn on [61]. But the introduction of this switching function has not a clear physical substantiation and can be justified a posteriori only.

Hence S-matrix becomes a functional of function $q(x)$ and the final state of system is expressed in the interaction representation as

$$\Phi[q] = S[q]\Phi_0, \tag{8.2}$$

Φ_0 is the initial state; the switching function is introduced into the (classical) action function, e.g.

$$S_{cl} = \int dx q(x)\pounds(x), \tag{8.3}$$

where $\pounds(x)$ is the Lagrange density of interaction. In the quantum field theory, correspondingly, the operator of evolution will be represented as the functional:

$$S[q] = T' \exp\left\{ i \int dx q(x)\pounds(x, q) \right\}, \tag{8.4}$$

T' is the chronologization operator and it is assumed that the relative value of Lagrangian depends on "intensity of interaction". This expression is the functional analog of the Schrödinger substitution used in Sect. 2.

The variation of (8.2) over $q(x)$ leads to the variational equation

$$i\delta\Phi[q]/\delta q(x) = \mathbf{H}(x; q)\Phi[q] \tag{8.5}$$

with the Hamiltonian of interaction

$$\mathbf{H}(x; q) = i(\delta S[q]/\delta q(x))S^*[q], \tag{8.6}$$

which is the variational analog, at $q = 1$, of the Schrödinger equation for S-operator in the interaction representation. This form leads to the covariant Tomonaga–Schwinger equation.

The switching function $q(x)$ describes the 4-volume of interaction, and if we shall *assume* that the extent of this region depends on details of interaction, we can rewrite (8.4) as

$$S[\pounds] = T' \exp\left\{ -i \int dx q(x, \pounds)\pounds(x) \right\} = T' \exp\left\{ -i \int dk q(-k, \pounds)\pounds(k) \right\}, \tag{8.7}$$

in the last equality the existence of corresponding Fourier transforms is proposed. The transition from (8.4) to (8.7) can be considered as the Legendre-type transformation $q \leftrightarrow \pounds$ of the classical action function (8.3), i.e. instead a switching of intensity of interaction, a variable part of the 4-volume of interaction (in particular, of the duration of interaction) is considered. This assumption is near to the procedure executed in Sect. 2 and can be partly substantiated below by argumentations of renormalization group method in Sect. 10.

The variation of (8.7) over $\pounds(k)$ leads to the equation

$$\delta S[\pounds]/i\delta\pounds(k) = q(-k, \pounds)S[\pounds], \tag{8.8}$$

or

$$q(-k, \pounds) = (\delta S[\pounds]/i\delta\pounds(k))S^{-1}[\pounds], \tag{8.8'}$$

i.e. to the evident variation-type analog of the temporal operator. Notice that in the complete accordance with the Bogoliubov method the singularity of £ on a hypersurface $\sigma(\omega)$ can be considered, which would lead to the equation

$$\delta S/i\delta\pounds(k,\sigma) = q(-k,\sigma)S(\sigma), \tag{8.9}$$

reciprocal to the Tomonaga–Schwinger equation.

These equations can be naturally brought to the equation of (1.5) type, reciprocal to the Schrödinger equation for S-matrix, with the formal temporal function

$$\tau(\omega) = \int dk q(-k,\pounds)(\delta\pounds(k)/\delta\omega). \tag{8.10}$$

The switching function $q(x)$ can be presented, in accordance with the adiabatic hypothesis (8.1), as

$$q(x) = \exp(-\gamma|t|/2) \quad \text{or} \quad q(-k) = \delta(k)/2\pi i(k_0 \pm i\gamma/2) \tag{8.11}$$

which can be rewritten in the covariant form with any unit time-like vector n_μ and replacement $t \to n_\mu x_\mu$. The substitution of (8.11) into (8.10) with assuming of δ-type properties of $\delta\pounds(k)/\delta\omega$ and frequency's shift $\omega_0 \to \omega - \omega_0$ leads to the usual form of temporal function for the simplest two-level system,

$$\tau \equiv \tau_1 + i\tau_2 = 1/\pi(\gamma/2 \pm i(\omega - \omega_0)). \tag{8.12}$$

Thus it can be concluded that the adiabatic hypothesis presents a non-obvious introduction of the time duration concept in theory.

9 Quantum Electrodynamics

Let's begin the consideration of temporal functions of QED with examination of the photon causal propagator of lowest order in vacuum (Feynman calibration, $\eta \to 0+$):

$$D_c(\omega,\mathbf{k}) = 4\pi/(\omega^2 - \mathbf{k}^2 + i\eta). \tag{9.1}$$

In accordance with all above it conducts to such expressions for time delay and duration of formation:

$$\tau_1 = -2\pi\omega\delta(\omega^2 - \mathbf{k}^2), \tag{9.2}$$

$$\tau_2 = 2\omega/(\omega^2 - \mathbf{k}^2) \sim 1/(\omega - |\mathbf{k}|). \tag{9.3}$$

The function τ_1 simply shows that the photon can be absorbed or emitted only completely. The function τ_2 qualitatively corresponds to the uncertainty principle, but is twice bigger, i.e. is measurable; it shows the possibility of retarded, at $\omega > |\mathbf{k}|$, or advanced, at $\omega < |\mathbf{k}|$, emission of photon.

It should be noted that the consideration of complete propagators through replacements $\mathbf{k}^2 \to \mathbf{k}^2 + P(k)$ in (9.1) with the polarization operator $P(k)$ of QED or even the transition to propagators of massive (scalar, for simplicity) particles does not change these general results.

The estimations of temporal values for elementary processes in the lowest orders can be achieved by such simple procedure: in accordance with (3.3) it can be suggested the expression for τ_2 via cross-section of scattering:

$$\tau_2 \sim -\frac{1}{2}(\partial/\partial\omega)\ln\sigma. \tag{9.4}$$

So, for the Rutherford scattering $\sigma \sim E^{-2}$and it gives $\tau_2 \sim 1/E$; for the nonrelativistic limit of Compton scattering $\sigma \sim (1-2\omega/m)$ and therefore $\tau_2 = 1/m(1-2\omega/m)$, etc. The values of τ_1 can be estimated now via dispersion relations (4.6) and so on.

The complete covariant generalization of temporal operator can be achieved by the Legendre transformation of equations for 4-momentum of interaction:

$$i\partial S/\partial x_\mu = k_\mu S \quad \longleftrightarrow \quad \partial S/i\partial k_\mu = x_\mu S, \tag{9.5}$$

where $x_\mu = (t, \mathbf{r})$ represents the 4-vector of "duration-space extent of interaction" in (2.10).

The temporal operator is now generalized as the covariant operator $\partial/i\partial p_\mu$, canonically conjugated to the energy-momentum operator $i\partial/\partial x_\mu$. The determination of corresponding functions can be established by the Ward–Takahashi identity:

$$\partial G/i\partial p_\mu = G(p)\Gamma_\mu(p, p; 0)G(p), \tag{9.6}$$

$G(p)$ is the particle Green function, $\Gamma_\mu(p, q; p-q)$ is the vertex part. Thus the expression for self-values of 4-operator follows:

$$\xi_\mu(p) \equiv \partial \ln G/i\partial p_\mu = \frac{1}{2}\{G(p)\Gamma_\mu(p, p; 0) + \Gamma_\mu(p, p; 0)G(p)\}. \tag{9.6$'$}$$

Notice that similar operators were introduced for localized states of spin zero massive particles [62], but they are a matter of discussions for photons [63].

The representation of vertex operator $\Gamma_\mu(p, p; 0) = \gamma_\mu - (\partial/\partial p_\mu)\Sigma$ with the mass operator Σ shows that (9.6) is connected with a gradual formation of physical particles.

The difference between both parts of ξ_μ can be visually demonstrated by consideration of the simplest case, the complete causal propagator $D_c = \overline{D} + D_1$ in the scope of scalar electrodynamics. In accordance with (9.6′) both parts of temporal function in the p-representation are equal to

$$\xi_{\mu 1}(p) \equiv \mathrm{Re}\,\xi_\mu(p) = p_\mu D_1(p; m) = p_\mu(D^{(+)} - D^{(-)}), \tag{9.7}$$

$$\xi_{\mu 2}(p) \equiv \mathrm{Im}\,\xi_\mu(p) = p_\mu \overline{D}(p; m) = p_\mu(D_{\mathrm{ret}} - D_{\mathrm{adv}}). \tag{9.7$'$}$$

In the x-representation these relations are even more descriptive:

$$\xi_{\mu 1}(x) = (\partial/\partial x_\mu)(D^{(+)}(x) - D^{(-)}(x)); \tag{9.8}$$

$$\xi_{\mu 2}(x) = (\partial/\partial x_\mu)(D_{\mathrm{ret}}(x) - D_{\mathrm{adv}}(x)), \tag{9.8$'$}$$

i.e. the duration of interaction describes a gradual decreasing of negative-frequency part and increasing of positive-frequency part, the extended duration of state formation is determined by difference of retarded and advanced parts alteration.

These results evidently show also the difference between uncertainty magnitudes and durations or space extents of interactions. So, the expression (9.7′) and $\overline{D}(p) = -\mathrm{P}\frac{1}{k^2}$ show that τ_2 and ρ_2 are approximately twice bigger corresponding uncertainty values.

Notice that on the base of these representations several particular models can be constructed. Let's consider as example the space extent of particle formation averaged over frequencies non exceeding the rest mass:

$$\langle\overline{\Delta}(\mathbf{r},m)\rangle=\frac{1}{m}\int_0^m d\mu\,\overline{\Delta}(\mathbf{r},\mu)=\sin(mr)/4\pi mr^2,$$

its gradient describes, via (9.8′), the space extent of interaction and it approaches, in accordance with the uncertainty principle, to $\delta(\mathbf{r})$ with increasing particle mass.

The temporal functions for electron must be determined via the electron Green functions and in the nearest order they are represented through (9.6′) as

$$\langle\tau(p)\rangle=\frac{1}{2}\,\mathrm{Tr}\{\gamma_0 S_{(\cdot)}(p)\}=p_0\Delta_{(\cdot)}(p), \tag{9.9}$$

which at the substitutions for $\omega\to(p_0^2-\mathbf{p}^2)^{1/2}$ and the Fourier transformation over moments variables coincides with (9.2), (9.3).

The physical sense of these functions can be established in such a way. The expression (9.8) shows that the temporal measurement is equivalent to adding zero-frequency scalar photon line into appropriate electron lines of the Feynman graphs. Therefore the durations can be interpreted via probed additional Coulomb fields of zero intensity (compare with the Baz' method of zero-intensity probe magnetic field and the "Larmor clock" in it [20, 21]).

This examination demonstrates, in particular, that the superluminal phenomena may be observable, in principle, in all scattering processes, not only in the QED.

In the spinor QED this 4-vector must be determined, correspondingly, as

$$\xi_\mu=\mathrm{Tr}\left(M^+\frac{\partial}{i\,\partial k_\mu}M\right)\Big/\,\mathrm{Tr}(M^+M). \tag{9.10}$$

It seems interesting to check by this expression the results, obtained in [9–12] for bremsstrahlung. By insertion of the known matrix element (e.g. [64]) into (9.10) it can be easy shown that $\mathrm{Re}\,\xi_\mu=0$ in the lowest order. It corresponds to the absence of any delay at bremsstrahlung, but the components of $\mathrm{Im}\,\xi_\mu$, connected to the formation processes, are nonzero. So, if ϵ, $\mathbf{k}$ and ϵ', $\mathbf{k}'$ are initial and final electron energies and momenta, ω is the photon energy, ϑ is the angle of electron departure, the duration and corresponding path extent of electron dressing are determined as

$$\tau_2=\frac{1}{\omega},\qquad \rho_2=\frac{k'}{\epsilon\omega}+\frac{1}{2}\frac{k'-k}{\epsilon\epsilon'+m^2} \tag{9.11}$$

at $\epsilon,\epsilon'\geq m$ and

$$\tau_2\approx|\rho_2|\sim\frac{2\epsilon(\epsilon'+\omega)}{m^2\omega},\qquad \rho_\perp\sim\frac{2\epsilon\vartheta}{m^2}, \tag{9.12}$$

when $\epsilon,\epsilon'\gg m$.

These results correspond to the previous calculations, but are obtained by a shorter and more general way. Notice that the region of photons formation can be considered as the near field of classical electrodynamics.

Let's briefly consider, as an example, some more general problems. So, if we investigate the scattering of scalar particles via one-particle exchange, the values of ξ_μ of the lowest

order are determined as logarithmic derivatives of intermediate particle propagator. In the standard description with taking into account the Ward–Takahashi identity it leads to the expression:

$$\xi_\mu(k) = (\partial / i\partial k_\mu) \ln D'_c = [2k_\mu / i(t - m^2)^2] \Gamma(t, t, 0) D'_c, \tag{9.13}$$

where D'_c is the complete Green function.

The factor $2k_\mu/(t - m^2)$ is formally close to the uncertainty principle and corresponds to the duration of outgoing particles formation $\tau_2 = |\rho_2| \sim 1/2E$. Time delay is connected with the imaginary part of propagator and arises at $t \geq 4m^2$, with possibilities of new particles birth.

As well as under photons formation the length of their formation (the near field region) appears, it can be *proposed* that in processes involving particles with additional internal parameters, another regions of their formation with their own peculiarities could be also manifested. Such possibilities will be briefly considering below.

10 To Interpretation of Some Renormalization Procedures

Let's begin with the Pauli–Villars method of regularization.

This method consists in the substitutions:

$$\Delta(p, m) \to \Delta(p, m) - \Delta(p, M) \sim \frac{(m^2 - M^2)}{(p^2 - m^2 + i\eta)(p^2 - M^2 + i\eta)} \tag{10.1}$$

with further passage to the limit $M \to \infty$.

What is its physical sense? Such substitution implies a decreasing of duration of new state formation with $p^2 < M^2$:

$$\xi_{\mu 2} \sim 2p_\mu \{(p^2 - m^2)^{-1} + (p^2 - M^2)^{-1}\}, \tag{10.2}$$

i.e. it is representing as a procedure of alteration of the interaction 4-volume, which was discussed in connection with the adiabatic hypothesis.

In the x-representation such substitution, $\xi_\mu(x) \to (\partial/\partial x_\mu)\{\Delta_c(x, m) + \Delta_c(x, M)\}$, leads to an increasing of the role of more energetic and more deep-seated virtual excitations at the beginning of calculations. Hence it actually means a partial account of higher terms of S-matrix in the process of particle formation.

Let's pass on to the subtraction procedures of renormalization.

The regularized mass function of the electron propagator is determined as

$$\Sigma^{\text{reg}}(p) = \Sigma(p) - \Sigma(p)|_{\gamma p = m} - (\gamma p - m)(\partial_p \Sigma(p))|_{\gamma p = m}. \tag{10.3}$$

Then the equalities

$$\Sigma^{\text{reg}}(p)|_{\gamma p = m} = 0, \qquad \partial_p \Sigma^{\text{reg}}(p)|_{\gamma p = m} = 0, \tag{10.4}$$

postulated at its renormalization, can be interpreted as two conditions: the mass of particle has definite magnitude and the process of its accumulation to the moment of regularization, at the infinity, is finished.

The regularized self-energetic part of the photon propagator ($k\partial_k \equiv k_\nu \partial/\partial k_\nu$)

$$\Pi^{\text{reg}}_{\mu\nu}(k) = \Pi_{\mu\nu}(k) - \left\{1 - k\partial_k - \frac{1}{2}(k\partial_k)^2\right\} \Pi_{\mu\nu}(k)|_{k=0}. \tag{10.5}$$

Apart from the evident and gauge equalities,

$$\Pi^{\mathrm{reg}}_{\mu\nu}(0)=0; \qquad k^{-2}\Pi^{\mathrm{reg}}_{\mu\nu}(k)|_{k=0}=0, \tag{10.6}$$

the conditions, that are usually simply postulated:

$$k\partial_k\Pi^{\mathrm{reg}}_{\mu\nu}(k)|_{k=0}=0, \qquad (k\partial_k)^2\Pi^{\mathrm{reg}}_{\mu\nu}(k)|_{k=0}=0, \tag{10.7}$$

must be physically interpreted as the conditions of the completeness of physical photon formation and impossibility of its self-acceleration.

Thus it is stated that the subtraction regularization corresponds to mathematical formulation of the common physical conditions primordially imposed on the system, and therefore these procedures are far from an artificial, ad hoc method.

It must be especially underlined that the method of renormalization group [62, 65, 66] can be reduced directly to the temporal functions. Really, as the corresponding Lie equations contain logarithmic derivatives of propagators over energy-momentum, they are still proportional to temporal magnitudes.

As the denominators of propagators leads only to the trivial terms, connected with the uncertainty principles or twice bigger them, let's consider for checking such proposition the nondimensional Green functions $\breve{G}(q)$ with all 4-momenta, except one, fixed. Then in accordance with the renorm-group equation of Callan and Symanzik [67–69] can be written that

$$|\xi_\mu|\sim q^2\frac{\partial}{\partial q^2}\ln\breve{G}(q)=(\gamma_m-1)\frac{\partial}{\partial m^2}\ln\breve{G}+\beta\frac{\partial}{\partial e}\ln\breve{G}-\gamma_G(m^2,e), \tag{10.8}$$

where γ_m, β and γ_G are the structure functions of the renorm-group.

In the lowest order of φ^4 theory $\gamma_m=0$, $\beta=\frac{3}{2}e^2$ and $\gamma_G=-\frac{3}{2}e$. Therefore in (10.9) for the 4-tail graphs are retained only the terms connected with a charge formation and the accumulation of observed mass:

$$\begin{aligned} &e\frac{\partial}{\partial e}\ln\breve{G}=e(1-\breve{G}^{-1});\\ &-\frac{\partial}{\partial m^2}\ln\breve{G}=-\frac{e^2}{m^2\breve{G}}\sum\frac{1}{y_k}\mathrm{Arth}\,y_k, \end{aligned} \tag{10.9}$$

where $y_k=(z_k^2-z_k)^{1/2}$, $(z_1,z_2,z_3)=(s,u,t)/m^2$.

It shows that in the φ^4 theory (and correspondingly in the QED) the charge increasing must extend the duration of formation, but in such gauge theories, where $\beta<0$, this process should decrease ξ_μ.

Note that in the UV limit $\gamma_m=1$ in (10.9) and

$$\ln\breve{G}(q^2,e)\to\ln\breve{G}(1,e)-2\gamma_G\ln q. \tag{10.10}$$

Hence in the asymptotically free theories, where $e\to-e$, the expression (10.9) can be reduced to such relation:

$$\xi_{\mu 2}(k)=\frac{2q_\mu}{q^2}\left(1-6\frac{|e|}{\nu}\right), \tag{10.11}$$

i.e. at $|e|=1/6\nu$ the duration of formation in this approximation is equal to zero.

This result can be of general interest in connection with number of lepton families and so on, but it requires more detailed further investigations. Note, in particular, that the absence of terms, which describe the delays in the expression (10.9), can be connected with the exclusively usage of one-loop approximation at calculation of matrix elements.

11 Conclusions

The main results of performed researches can be formulated in such points.

1. The Wigner–Smith function of time delay in scattering process and the function of duration of state formation are combined into the temporal (analytic) function. This function can be represented via differential equation in p-space for S-matrix or other response function, which is reciprocal to the Schrödinger equation. At given temporal function this equation can be used for calculation or modeling of S-matrix elements and so on.
2. The magnitudes of duration of scattering process (time of delay and duration of state formation) are implicitly contained in the usual field theory. They actually correspond to the propagators of interacting fields and thereby many problems of kinetics could be considered without explicit introducing of temporal magnitudes. Just it can explain a more recent beginning of researches of temporal parameters in the quantum theory.
3. The adiabatic hypothesis of quantum theory represents an implicit expression of existence of the certain duration of formation (dressing) of physical particles. Therefore it does not represent a formal, pure mathematical procedure, but shows that at the investigation of arising of physical state the proper consideration of its formation duration can be essential.
4. Both methods of duration measurement, by Wigner and Smith and by "Larmor clock", can be described as an addition of zero-energy scalar line to the Feynman graph of process.
5. The dispersion relations for temporal functions are established. They prove, in particular, that the duration of state formation is, at least, twice bigger the uncertainty values and therefore is measurable. Such magnitudes can be directly measured in the multiphoton processes, etc.
6. The consideration of the Lorentz (oscillator) model of simple dispersive medium leads to an intuitively evident interpretation of temporal functions. In this model the function of time delay is proportional to polarization of medium and the function of state formation is proportional to electric conductivity.
7. The transition from the Schrödinger equation into the reciprocal temporal equation corresponds to the Legendre transformation of classical action function. The covariant form of temporal equation is deduced by a temporal variant of the Stueckelberg–Bogoliubov variational method.
8. The methods of subtraction regularization in field theory can be logically justified and explained as the primordial imposing of such physical requirements on propagators of particles as asymptotic finishing of their formation and impossibility of particles self-acceleration.
9. The concept of interactions duration imparts the evident physical sense to the equations of renormalization group and demonstrates that the formation of each particles parameters required the certain (may be, specific) duration.

Hence it gives a possibility to think that the coordination of durations of these partial processes would reveal some peculiarities of those or other particles. Such program requires, however, further researches.

We do not discuss here a lot of delay determinations known in the current literature. It can be suggested that the revealed analytical properties of the composed temporal functions demonstrate their general significance. It does not exclude, of course, the possible usefulness of some other determinations in special cases.

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